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POLYTROPES IN PHASE PLANE

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Abstract— Polytropes are gaseous spheres in thermodynamic and hydrostatic equilibrium with a certain equation of state that are very useful in realistic stellar modal. In this paper we have demonstrated that the pressure of the stars keeps on increasing from its surface to centre. Approximate analytic solutions to the equilibrium equations have been presented in phase planes such as (Up,Vp), Transformations connecting solutions in this phase plane have been obtained and discussed.

Index Terms— Polytropes, stellar modal, Analytical Study, thermodynamic and hydrostatic equilibrium.

1 INTRODUCTION

Lane-Emden equation is a non-linear second order differential equation which governs the structure of a polytropic gas sphere in equilibrium under its own gravitation .The equation is of importance in astrophysics because, for the valve of the polytropic index n between 0 and 3, the equation approximates to a reasonable accuracy the structure of a variety of the realistic stellar models. though modern texts no longer give them through treatment that classical works of Emden and Chandrasekhar do, Closed form analytical solution, has been studied by Fowler[1], Hopf[2] and Chandrasekhar for n<3, n=3, and n>3, respectively. It is well known so far from some of these studies that the polytropic index n=0 and 1 represent, the liquid and gaseous states of a polytrope of uniform density respectively. The origin and the behavior of Lane-Emden equations were reported same whatever be the index of a polytrope[3-14]. The Miline[3] was able to determine the maximum limiting density[15], whereas the structure of planet was also reported[16, 17] for the same values of n, and the maximum value of mass of a star[18] for n→0 and n→1 general relativity neutron star[19] were also reported for the same values of n, and very massive stellar models in Ni's theory of gravity[20] , relativistic stellar structures and X-ray transients in Ni's theory of gravity[21], Further thermo dynamical equilibrium of stars clusters embedded in an isothermal configuration[22].

Considering the stars, which are in equilibrium and in a steady state can be characterized by three physical parameters i.e. its mass *M*; its radius *R*; and its luminosity *L*(*L* refers to the amount of radiant energy in ergs , radiated by the star per second to the space outside,) analytic series solutions to the equilibrium equations have been presented in phase planes such as (Up,Vp), Transformations connecting solutions in this phase plane have been obtained, Since the nucleus includes the immediate neighborhood of the origin (n=0), it will be of the interest to investigate it, in the light of this new concept of uniform density for n →0 and n →1.

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2 ARRANGEMENT IN (V_P, U_P) PHASE PLANE :-

The equations governing the structure of a polytopic configuration of index n with angular velocity Ω can be expressed with the help of electromagnetic Maxwall's equations

$$\frac{P}{\rho}=\nabla\phi+\frac{1}{2}\Omega^2X^2,\ X^2=x^2+y^2 \quad (1)$$

$$P=K\rho^{1+\frac{1}{n}} \quad (2)$$

$$\nabla^2\varphi=-4\pi G\rho \quad (3)$$

where, P is the pressure, ρ the density, ϕ the gravitational potential, X the distance from the axis of rotation, K a constant, and G the gravitational constant (6.67 x 10^{-8} dynes cm^2/gm^2).

Equation (1), (2) and (3) enable us to write the generalized equation in (ξ_P, P) plane in the form.

$$\frac{1}{\xi_P^N}\frac{d}{d\xi_p}\left(\xi_P^N P^{-\left(\frac{n}{n+1}\right)}\frac{dP}{d\xi_p}\right)=-P^{\frac{n}{n+1}}+\sigma\omega \quad (4)$$

where $r\equiv\alpha_P\xi_P$ & $\left.\rho=\rho_c\theta^n,\sigma=\rho_c K^{\frac{n}{n+1}}\right\}$ (4a)

$$\alpha_p=\frac{k^{\frac{n}{n+1}}}{2\sqrt{\Pi G}}$$

for non-rotating case ω= 0 and Polytropic index n=1 Equation (4) becomes :

$$\frac{1}{\xi_P^N}\frac{d}{d\xi_p}\left(\xi_P^N\ P^{\frac{-1}{2}}\frac{dP}{d\xi_p}\right)=-P^{\frac{1}{2}} \quad (5)$$

$$\text{or, } \frac{N}{\xi_P}\frac{dP}{d\xi_p}-\frac{1}{2P}\left(\frac{dP}{d\xi_p}\right)^2+\frac{d^2P}{d\xi_P^2}=-P \quad (6)$$

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Now we define Up and Vp for non-roating case i.e. ω= 0 and for polytropic index n=1, as follows

$$Up = -\frac{\xi_p P}{P'} \tag{7a}$$

$$Vp = -\frac{\xi_p P'}{P} \tag{7b}$$

Where $P' = \frac{dP}{d\xi_p}$ are homology invariant functions.

The First order equation between U_p, & V_p can be obtained as follows,

From equation (4a)

$$\frac{1}{U_p}\frac{dU_p}{d\xi_p} = \frac{1}{\xi_p} + \frac{P'}{P} - \frac{P''}{P'} \tag{8}$$

from (6), we can write.

$$P'' = -P - \frac{N}{\xi_p}P' + \frac{1}{2P}(P')^2 \tag{9}$$

where,

$$P' = \frac{dp}{d\xi_P} \ and \ p''\frac{d^2p}{d\xi_p^2}$$

Using (8) & (9) we get,

$$\frac{1}{U_p}\frac{dU_p}{d\xi_p} = \frac{1}{\xi_P}\left[1 + \frac{\xi_P P'}{P} - \frac{\xi_P P''}{P'}\right]$$

$$= \frac{1}{\xi_p}\left[1 + \frac{\xi_p p'}{p} - \frac{\xi_p}{P'}\left(-p - \frac{N}{\xi_p}P' + \frac{1}{2p}(p')^2\right)\right]$$

$$= \frac{1}{\xi}\left[(1+N) - \frac{1}{2}\frac{\xi_p p'}{p} + \frac{\xi_p p}{p'}\right]$$

Using equation (7a), we get

$$\frac{1}{U_p}\frac{dU_p}{d\xi_p} = \frac{1}{\xi_p}\left[(1+N) - \frac{Vp}{2} - Up\right] \tag{10}$$

From equation (7b), we get.

$$\frac{1}{Vp}\frac{dV_p}{d\xi_p} = \frac{1}{\xi_p} - \frac{P'}{P} + \frac{P''}{P'} \tag{11}$$

Using (11) and (9), and after simplification, by above process, we get,

$$\frac{1}{V_p}\frac{dV_p}{d\xi_p} = \frac{1}{\xi_p}\left[1 - N + U_p + \frac{1}{2}V_p\right] \tag{12}$$

Using Equation (10) & (12) we get,

$$\frac{V_p}{U_p}\frac{dU_p}{dV_p} = -\left[\frac{U_p + \frac{1}{2}V_p - (1+N)}{U_p + \frac{1}{2}V_p + (1-N)}\right]$$

or

$$\frac{dU_p}{dV_p} = -\frac{U_p}{V_p}\left[\frac{U_p + \frac{1}{2}V_p - (1+N)}{U_p + \frac{1}{2}V_p + (1-N)}\right] \tag{13}$$

This is the required structure equation in (Vp, Up) phase plane.

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2.1 Solution of structure equation in (V_p, U_p) phase plane :-

We solve the equation (13) for three cases, spheroidal (N=3), cylindrical (N=1), and for plane symmetrical case (N=0).

We solve the structure equation (13) by assuming series solution as followes,

consider,

$$U_p = (1+N) + a_1V_p + a_2V_p^2 + a_3V_p^3 + a_4V_p^4 + ... \tag{14}$$

We find that it satisfies the initial condition

$$U_p \to (1+N) \ \& V_p \to 0 \quad as \ \xi_p \to 0$$

differentiating equation (14) w.r.t V_p, we get

$$\frac{dU_p}{dV_p} = a_1 + 2a_2V_p + 3a_3V_p^2 + 4a_4V_p^3 + 5a_5V_p^4 + ... \tag{15}$$

putting the value of U_p & $\frac{dU_p}{dV_p}$ (14) & (15) respectively, in (13), we get,

$$\Rightarrow V_p\left(a_1 + 2a_2V_p + 3a_3V_p^2 + 4a_4V_p^3 + ...\right)\left[(1+N) + \left(a_1 + \frac{1}{2}\right)V_p\right.$$

$$\left. + a_2V_P^2 + a_3V_p^3 + a_4V_p^4 + ...(1+N)\right]$$

$$= -\left[\quad (1+N) + a_1V_p + a_2V_p^2 + a_3V_p^3 + a_4V_p^4 + ...\right.$$

$$\left.\left\{(1+N) + \left(a_1 + \frac{1}{2}\right)V_p + a_2V_p^2 + a_3V_p^3 + a_4V_p^4 + ... - (1+N)\right\}\right]$$

$$\Rightarrow \left[\left(a_1V_p + 2a_2V_p^2 + 3a_3V_p^3 + 4a_4V_p^4 + ...\right)\left(2 + \left(a_1 + \frac{1}{2}\right)V_p + a_2V_p^2 + a_3V_p^3 + a_4V_p^4 + ...\right)\right]$$

$$= -\left[\left\{(1+N) + a_1V_p + a_2V_p^2 + a_3V_p^3 + a_4V_p^4 + ...\right\}\right.$$

$$\left.\left\{\left(a_1 + \frac{1}{2}\right)V_p + a_2V_p^2 + a_3V_p^3 + a_4V_p^4 + ...\right\}\right]$$

$$\Rightarrow 2a_1V_p + \left\{a_1\left(a_1 + \frac{1}{2}\right) + 4a_2\right\}V_p^2 + \left\{a_1a_2 + 2a_2\left(a_1 + \frac{1}{2}\right)\right.$$

$$+6a_3\Big\} V_p^3 + \Big\{(a_1a_3 + 8a_4 + 3a_3\Big(a_1 + \frac{1}{2}\Big) + 2a_2^2\Big\}V_p^4 ....$$

$$= -\Big[(1+N) + \Big(a_1 + \frac{1}{2}\Big)V_p + \Big\{a_2\Big(a_1 + \frac{1}{2}\Big) + (1+N)a_2\Big\}V_p^2\Big]$$

$$+\Big\{(1+N)a_3 + a_1a_2 + a_2\Big(a_1 + \frac{1}{2}\Big)\Big\}V_p^3 + \{(1+N)a_4$$

$$+a_1a_3 + a_2^2 + a_3\Big(a_1 + \frac{1}{2}\Big)\Big\}V_p^4 + ...\Big] \qquad (16)$$

Now we disscuss about equation (16) in three different cases :

2.2 :- For Spheroidal shape : (N=2).

from Equation (16),

$$2a_1V_p + \Big\{a_1\Big(a_1 + \frac{1}{2}\Big) + 4a_2\Big\}V_p^2 + \Big\{a_2a_2 + 2a_2\Big(a_2 + \frac{1}{2}\Big) + 6a_3\Big\}V_p^3 +$$

$$\Big\{a_1a_3 + 8a_4 + 3a_3\Big(a_1 + \frac{1}{2}\Big)\Big\} + 2a_2^2\ V_p^4 + ...$$

$$= \Big[3\Big(a_1 + \frac{1}{2}\Big)V_p + \Big\{a_1\Big(a_1 + \frac{1}{2}\Big) + 3a_2\Big\}V_p^2 + \Big\{3a_3 + a_1a_2 + a_2\Big(a_1 + \frac{1}{2}\Big)\Big\}V_p^3 + \Big\{3a_4 + a_1a_3 + a_2^2 + a_3\Big(a_1 + \frac{1}{2}\Big)\Big\}V_p^4\Big]$$

Equating the co-efficients of powers of V_p, we get

$$2a_1 = -3\Big(a_1 + \frac{1}{2}\Big) \Rightarrow a_1 = \frac{-3}{10}$$

$$a_1\Big(a_1 + \frac{1}{2}\Big) + 4a_2 = -\Big\{a_1\Big(a_1 + \frac{1}{2}\Big) + 3a_2\Big\}\Big]$$

$$2a_1\Big(a_1 + \frac{1}{2}\Big) = -7a_2$$

$$\Rightarrow a_2 = \frac{3}{175}$$

$$a_1a_2 + 2a_2\Big(a_1 + \frac{1}{2}\Big) + 6a_3 = -3a_3 - a_1a_2 - a_2 = \Big(a_1\frac{1}{2}\Big)$$

$$\Rightarrow a_3 = 0$$

$$a_1a_3 + 8a_4 + 3a_3\Big(a_1 + \frac{1}{2}\Big) + 2a_2^2$$

$$= -3a_4 - a_1a_3 - a_2^2 - a_3\Big(\alpha_1 + \frac{1}{2}\Big)$$

$$11a_4 = -3a_2^2$$

$$\Rightarrow a_4 = -\frac{27}{336875}$$

$\therefore$ for N = 2, series solution becomes, from equation (14)

$$U_p = 3 - \frac{3}{10}V_p + \frac{3}{175}V_p^2 - \frac{27}{336875}V_p^4 + ... \qquad (17)$$

2.3 For Cylindrical shape i.e. N=1.

From equation , (16)

$$2a_1V_p + \Big\{a_1\Big(a_1 + \frac{1}{2}\Big) + 4a_2\Big\}V_p^2 + \Big\{a_1a_2 + 2a_2\Big(a_1 + \frac{1}{2}\Big) + 6a_3\Big\}$$

$$V_p^3 + \Big\{a_1a_3 + 8a_4 + 3a_3\Big(a_1 + \frac{1}{2}\Big) + 2a_2^2\Big\}V_p^4 + ...$$

$$= -\Big[2\Big(a_1 + \frac{1}{2}\Big)V_p + \Big\{a_1\Big(a_1 + \frac{1}{2}\Big) + 2a_2\Big\}V_p^2 +$$

$$\Big\{2a_2 + a_1a_3 + a\Big(a_1 + \frac{1}{2}\Big)\Big\}V_p^3 +$$

$$\Big\{2a_4 + a_1a_3 + a_2^2 + a_3\Big(a_1 + \frac{1}{2}\Big)\Big\}V_p^4 + ...\Big]$$

Equating the co-efficient of powers of V_P we get,

$$2a_1 = -2\Big(a_1 + \frac{1}{2}\Big)$$

$$\Rightarrow a_1 = -\frac{1}{4}$$

$$a_1\Big(a_1 + \frac{1}{2}\Big) + 4a_2 = -a_1\Big(a_1 + \frac{1}{2}\Big) - 2a_2$$

$$\Rightarrow a_2 = -\frac{1}{48}$$

$$a_1a_2 + 2a_2\Big(a_1 + \frac{1}{2}\Big) + 6a_3$$

$$= -2a_3 - a_1a_2 - a_2\Big(a_1 + \frac{1}{2}\Big)$$

$$\Rightarrow a_3 = -\frac{1}{1536}$$

$$a_1a_3 + 8a_4 + 3a_3\Big(a_1 + \frac{1}{2}\Big) + 2a_2^2$$

$$= -2a_4 - a_1a_3 - a_2^2 - a_3\Big(a_1 + \frac{1}{2}\Big)$$

$$a_4 = -\frac{3}{30720}$$

$\therefore$ For N=1, series solution becomes, from equation (14).

$$V_p = 2 - \frac{1}{4}V_p + \frac{1}{48}V_p^2 - \frac{1}{1536}V_p^3 - \frac{3}{30720}V_p^4 + ... \qquad (18)$$

2.4 : For plane - symmetric shape i.e. N = 0 from equation (16)

$$2a_1 V_p + \left\{a_1\left(a_1 \frac{1}{2}\right) + 4a_2\right\} V_p^2 + \left\{a_1 a_2 + 2a_2\left(a_1 + \frac{1}{2}\right)\right.$$

$$\left. + 6a_3\right\} V_p^3 + \left\{a_1 a_3 + 8a_4 + 3a_3\left(a_1 + \frac{1}{2}\right) + 2a_2^2\right\} V_p^4 + ...$$

$$= -\left(a_1 + \frac{1}{2}\right) V_p - \left\{a_1\left(a_1 + \frac{1}{2}\right) + a_2\right\} V_p^2$$

$$+\left\{a_3 + a_1 a_2 + a_2\left(a_1 + \frac{1}{2}\right)\right\} V_p^3 + \left\{a_4 + a_1 a_3 + a_2^2 + a_3\left(a_1 + \frac{1}{2}\right)\right\} V_p^4 + ...$$

Equating the co-efficient of powers of V_p

$$2a_1 = -\left(a_1 + \frac{1}{2}\right) \Rightarrow a_1 = -\frac{1}{6}$$

$$a_1\left(a_1 + \frac{1}{2}\right) + 4a_2 = -a_1\left(a_1 + \frac{1}{2}\right) - a_2$$

$$\Rightarrow a_2 = \frac{1}{45}$$

$$a_1 a_2 + 2a_2\left(a_1 + \frac{1}{2}\right) + 6a_3 = -a_3 - a_1 a_2 - a_2\left(a_1 + \frac{1}{2}\right)$$

$$\Rightarrow a_3 = -\frac{2}{945}$$

$$a_1 a_3 + 8a_4 + 3a_3\left(a_1 + \frac{1}{2}\right) + 2a_2^2$$

$$9a_4 = -2a_1 a_3 - 3a_2^2 - 4a_3\left(a_1 + \frac{1}{2}\right)$$

$$\Rightarrow a_4 = \frac{1}{14175}$$

$\therefore$ For N=0, from equation (14), the series solution becomes :

$$U_p = 1 - \frac{1}{6} V_p + \frac{1}{45} V_p^2 - \frac{2}{945} V_p^3 + \frac{1}{14175} V_p^4 + ... \qquad (19)$$

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2.4 Results and Discussion

graphical representation of (Up,Vp) phase plane for N =2 & n=1 (Fig. 1) , for N =1 & n=1 (Fig.2) and N=0 & n=1 (Fig. 3), where Up show pressure and Vp show radius of polytropes. The graphs plotted by our series solution method are in good agreement by the graph with the stellar model[19]. It is evident from the figure that the as pressure of the polytropes increases, its radius decreases in all the three cases implying that the mass of the stars keeps on increasing as we move from surface to centre. The graph for N=0 , N=1, and N=2 between Up & Vp has been plotted and found to be in good agreement with the results graph of N=0 (plane Symmetric) N=1 (Cylindrical) N=2(spheroidal) the shape stellar structure of given value.

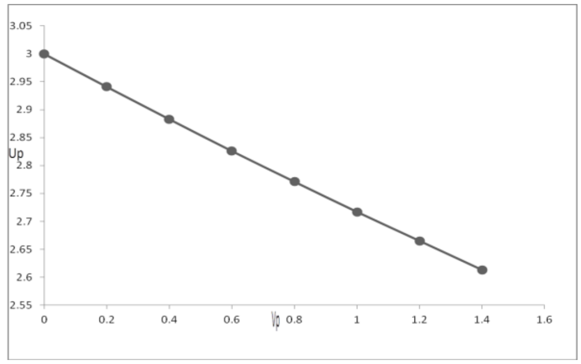

Fig 1. Graphical representation of (Vp,Up) phase plane for N =2 & n=1 where Vp show radius ,Up show pressure of polytropes.

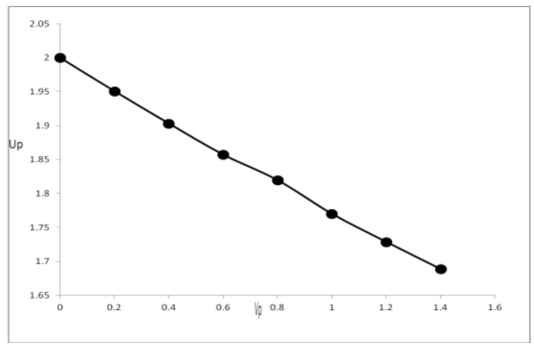

Fig 2. Graphical representation of (Vp,Up) phase plane for N =1 & n=1 where Vp show radius ,Up show pressure of polytropes.

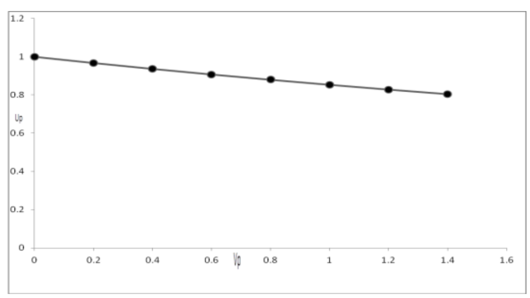

Fig 3. Graphical representation of (Vp,Up) phase plane for N =0 & n=1 where Vp show radius ,Up show pressure of polytropes.

4 Conclusion

An unified analytic study structure of the nucleons of Polytropes N=0 (Plane Symmetric) N=1 (Cylindrical) N=2 (spheroidal) has been investigated following the concept of sphere of uniform density defined by polytropic index (n) tending to zero. The graphs plotted by our series solution method are in good agreement by the graph with the stellar mod-

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el[19]. The mass of the stars keeps on decreasing as we move from centre to surface. Our given analysis can be applied to the interdisciplinary modeling, environmental and biological systems which may quite often involve complicated forms of linear or non-linear differential equation.

REFERENCES

[1] R.H. Fowler and Grid, "Properties of E solution an $\xi \to 0$" MNRAS, Vol .91,pp .63-71 1930.

[2] E .Hopf ," On Solution of Lane-Emden Equation for $1<n \leq 3$ "MNRAS,Vol. 91, pp.635-662,1932 .

[3] E.A. Miline "Structure of self-gravitating polytropic sphere",Hdb D ap,Vol.3,pp 65-71,1930.

[4] J.P. Sharma " Relativistic stellar Structures and X-ray Transients in Ni's theory of Gravity" Nat Acad Sc India ,2006..

[5] Indira . Bradoloi Ball , Astr India ,pp.32, 2004.

[6] J.P. Sharma Proch: Math . Soc. BHU,vol. 16,2000.

[7] J .P. Sharma "Thesis of D. Sc, Univ of Gorakhpur" (U.P.) 1981.

[8] J.P Sharma. "Thesis of D. Phil , Univ of Allahabad" (U.P.) 1970.

[9] E.R. Harrison and R.G. Lake, Ap.J, pp, 171- 323. 1972.

[10] J.Ostriker Ap.J : pp 140 ,1964.

[11] Shambhunath .Srivastava and, J. P Sharma. Experientia ,vol, 24: pp 764, 1968.

[12] D Menzel. , P.L Bhatnagar.and , H.K Sen." Stellar Interiors" Chapman and Hall, London, chap – 10,1963.

[13] S. Chandrasekhar "An Introduction to the study stellar structure" Dover Publication, Inc, N.Y. 1939.

[14] J. Homer Lane. American Journal of Science,2 dser. , pp 50-57,1869.

[15] J.P Sharma. "Limiting density in white dwarfs", Indian J Phys vol.48, pp. 66-70.1974.

[16] R.A. Lyttleton "On the Origin of Mountains" Proc Roy Soc vol. 275, pp 1-11.1963.

[17] J.P. Sharma " Planetary structures in general relativity" PA-GEOPH,vol.97: p 14-24 1972.

[18] J.P. Sharma " On the maximum value of the mass of star" Trans New York Acad Sc. Vol,34: pp 691-694. 1972.

[19] Sanish Thomas and J. P. Sharma. "General relativistic Neutron stars " Pramana, 2009.

[20] J.P. Sharma. D.P Jayapandian and Sanish Thomas " Very Massive stellar models in Ni's theory of gravity" proc Nat Acad Sc vol.79, pp 411- 416 ,2009.

[21] J.P. Sharma" Relativistic Stellar structure and X-rays Transients in Ni's theory of gravit", Proc Nat Acad Sc Vol, 79 . pp 29-34 2006

[22] J.P. Sharma" Thermodynamically equilibrium of star cluster" Proc Nat Acad Sc Vol.76 . pp 305-307,2006.